Continuous Models and Normal Probabilities

Lecture 14

QBA201 – Quantitative Business Analysis

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Continuous probability is assigned to intervals

For a variable with a continuous density, probability is area under the density curve.

We can ask for the probability that a task takes between four and six minutes.

A density height is not itself a point probability.

An exact point has probability zero

For a continuous density model:

$$\mathrm{P}(X = 5) = 0.$$

But an interval around five can have positive probability.

$$\mathrm{P}(4.9 < X < 5.1) > 0$$

when the density is positive there.

Two checks for a density

$$f(x) \geq 0, \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

For endpoints $a < b$:

$$\mathrm{P}(a < X < b) = \int_a^b f(x)\,dx.$$

The total area is one. A density height can exceed one without violating this rule.

Uniform density has a constant height

For X uniform between a and b:

$$f(x) = \frac{1}{b-a} \quad (a \le x \le b),$$

and zero outside that range.

$$\text{Total area} = (b-a)\frac{1}{b-a} = 1.$$

Discuss: Why does a wider range require a lower density?

Translate the uniform-sales question

Daily sales X are uniform from 2,000 to 5,000 litres.

$$f(x) = \frac{1}{3,000}.$$

Find the probability of sales between 2,500 and 3,000 litres:

$$\mathrm{P}(2,500 \leq X \leq 3,000).$$

Uniform probability is width times height

$$\mathrm{P}(2,500 \leq X \leq 3,000) = (3,000 - 2,500)\frac{1}{3,000} = \frac{1}{6} = 0.1667.$$

Interpretation

About 16.67% of days under this uniform model lie in the requested range.

Discuss: Why is the interval width 500 rather than 3,000?

Calculate the uniform upper tail

$$\mathrm{P}(X \geq 4,000) = \frac{5,000 - 4,000}{5,000 - 2,000} = \frac{1}{3} = 0.3333.$$

Check by complement:

$$1 - \mathrm{P}(X < 4,000) = 1 - \frac{2,000}{3,000} = \frac{1}{3}.$$

Rounding creates a measurement interval

If sales are rounded to the nearest litre, a recorded value of 2,500 corresponds approximately to:

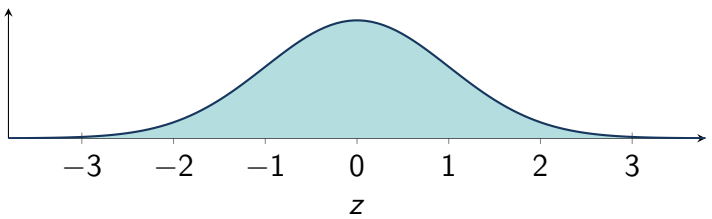
$$2,499.5 \leq X < 2,500.5.$$

Under the uniform model its probability is:

$$1/3,000 \approx 0.000333.$$

Exact model values and rounded recorded values answer different questions.

The normal model is symmetric and bell-shaped



A normal variable has mean μ and standard deviation $\sigma > 0$. The center is μ and the total area is one.

The standard normal has fixed center and spread

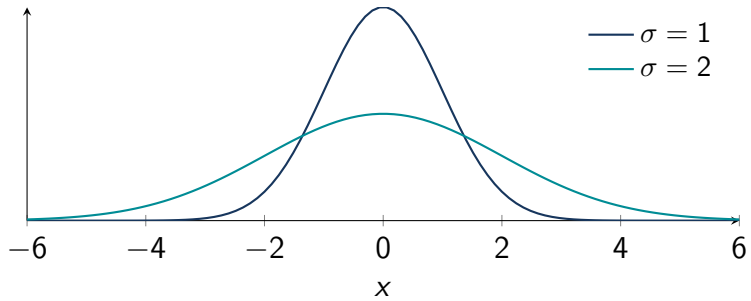
$$Z \sim N(0,1), \qquad \mathrm{E}(Z) = 0, \quad SD(Z) = 1.$$

Its cumulative distribution is:

$$\Phi(z) = \mathrm{P}(Z \leq z).$$

We will express other normal probabilities using this one distribution.

Changing the SD changes spread and peak height



Both curves have mean zero and area one.

Standardization uses distance in SD units

If X is normal with mean μ and SD σ:

$$Z = \frac{X - \mu}{\sigma}.$$

For a cutoff x, calculate:

$$z = \frac{x - \mu}{\sigma}.$$

Standardize the cutoff as well as the random variable.

Dividing by the SD changes the measurement scale

$$Z = \frac{X - \mu}{\sigma}$$

has mean zero and SD one.

Normal standardization preserves the normal family. It expresses the horizontal axis in standard-deviation units.

Standardizing an arbitrary nonnormal variable does not make it normal.

Petrol demand: define the random variable

Let X be today's demand in litres.

$$\mu = 1,000, \qquad \sigma = 100.$$

Assume demand is normal and available stock is 1,100 litres.

Discuss: Which event means the stock is sufficient?

Enough stock means demand does not exceed stock

Target: $\mathrm{P}(X \leq 1,100)$.

This is the area to the left of 1,100.

Stock shortage is the complementary event $X > 1,100$.

Translate the business statement before choosing the tail.

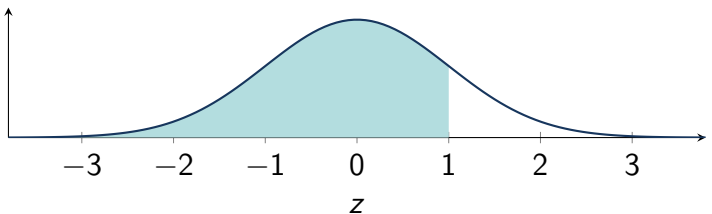
Standardize the demand cutoff

$$z = \frac{1,100 - 1,000}{100} = 1.$$

Because division is by a positive SD, the inequality direction is unchanged:

$$\mathrm{P}(X \leq 1,100) = \mathrm{P}(Z \leq 1).$$

The required normal area



$P(Z \leq 1) = \Phi(1)$. $P(\text{enough stock}) = 0.8413$.

Read the row and column together

For $z = 1.52$, use row 1.5 and column .02.

$$\Phi(1.52) = 0.9357.$$

For $z = 1.00$, use row 1.0 and column .00.

$$\Phi(1.00) = 0.8413.$$

First check whether your table reports left-tail probabilities or a different area.

A negative z-score uses symmetry

$$\Phi(-1.52) = 1 - \Phi(1.52).$$
$$= 1 - 0.9357 = 0.0643.$$

The area lies in the lower tail.

A negative z-score has a positive probability.

An upper-tail question needs a complement

$$\begin{aligned} \mathrm{P}(Z > 1.80) &= 1 - \Phi(1.80). \\ &= 1 - 0.9641 = 0.0359. \end{aligned}$$

Discuss: Why would reporting 0.9641 answer the opposite tail?

Between two values means a difference of cumulatives

$$\mathrm{P}(-1.30 < Z < 2.10) = \Phi(2.10) - \Phi(-1.30).$$

Using the supplied table:

$$0.9821 - 0.0968 = 0.8853.$$

Subtract the lower cutoff's cumulative area from the upper cutoff's area.

An unlisted tail is not exactly zero

$$\mathrm{P}(Z > 3.10) \approx 0.000968.$$

A short printed table may not show all useful digits.

Report a sufficiently small bound or use software when tail accuracy matters.

A normal-return illustration

The next example compares the probability of a negative return under two hypothetical normal models.

It illustrates how dispersion affects a specified tail probability.

A normal model is an assumption, not a guarantee about real investment returns.

Define loss before calculating its probability

Suppose a return R is normal with mean 10% and SD 5%.

Losing money means $R < 0$.

Discuss: Should you use 0, 5% or 10% as the event's cutoff?

Read the loss probability

Standardize the zero-return cutoff: $z = (0 - 0.10)/0.05 = -2$.

$$\mathrm{P}(R < 0) = \Phi(-2) = 0.0228.$$

The loss probability is about 2.28% under this particular model.

Discuss: Does a positive expected return rule out a loss?

More spread can increase the chance of loss

Keep mean 10%, but raise SD to 10%.

$$z = \frac{0 - 0.10}{0.10} = -1.$$

$$\mathrm{P}(R < 0) = \Phi(-1) = 0.1587.$$

With the same positive mean, the zero-return cutoff is now closer in SD units.

An inverse question starts with an area

Define z_α by an *upper-tail* probability:

$$\mathrm{P}(Z > z_\alpha) = \alpha.$$

Then:

$$\Phi(z_\alpha) = 1 - \alpha.$$

Discuss: Which cumulative probability should you look up for an upper-tail area of 0.025?

A 2.5% upper tail gives 1.96

$$1 - \alpha = 1 - 0.025 = 0.975.$$

$$\Phi(1.96) \approx 0.975, \qquad z_{0.025} \approx 1.96.$$

For a general normal variable, the corresponding cutoff is:

$$x = \mu + \sigma z.$$

Review: score, probability and percentile

1. Let Y be normal with mean 720 and SD 110. At 880, $z = (880 - 720)/110 \approx 1.45$.
2. Using the supplied rounded-table values, $\mathrm{P}(650 < Y < 900) \approx 0.94950 - 0.26109 = 0.68841$.
3. For a new normal model with $\mu = 350, \sigma = 60$, the 90th percentile is $350 + 60(1.2816) \approx 426.90$.